A Complete Solution to the
Black-Scholes Option Pricing Formula*

Ravi Shukla
Whitman School of Management
Syracuse University
Syracuse NY, 13244
315-443-3576
rkshukla@syr.edu

Michael J Tomas, III
Assistant Professor of Finance
School of Business Administration
University of Vermont
Burlington, VT 05405
802-656-8270
MTOMAS@bsad.uvm.edu

September 2006

*Please do not quote. Comments are welcome.
We are grateful to Raj Aggarwal, Fernando Diaz, Tom Finucane, and Mark Holder for providing valuable feedback.
A Complete Solution to the Black-Scholes Option Pricing Formula

Abstract

Option pricing theory is one of the major contributions of finance academics in the last thirty years. The early work of Black and Scholes (1972) is now legendary. Nearly every textbook on option theory covers this work. The textbooks derive the partial differential equation and present the solution to the equation but do not provide the mathematical details of the solution to the partial differential equation. In this paper we present a complete solution that can be used as a pedagogical aid in classes on derivatives, options and futures, or financial engineering.
A Complete Solution to the Black-Scholes Option Pricing Formula

1 Introduction

There has been a significant growth in the popularity of courses on derivatives at college campuses. A major factor in this growth has been an increasing number of specialized programs in financial engineering, mathematical finance and computational finance. As of June 2000, there were at least 37 such programs. There has also been an increase in the use of derivative securities by market participants in recent years. This attention to derivatives in the academic as well as professional arena has resulted in a variety of texts discussing the theory and practice of pricing these instruments. Cox and Rubinstein (1985), Hull (2000), Jarrow and Turnbull (2000) and Neftci (2000) are representative examples.

Every textbook discusses the development of the enormously important work of Black and Scholes (1972) since most option pricing models are based on this work. The presentation, almost universally, consists of using arbitrage arguments to derive the partial differential equation (PDE) and then presenting the solution to the PDE under the appropriate initial and boundary conditions. The textbooks and even the original Black-Scholes work, however, do not actually derive the solution to the PDE. They make references to books on differential equations which provide the steps necessary to solve the equation. These books also do not contain the direct solution to the PDE. The student is left to construct the solution on his/her own.

One text, Shimko (1992), does present many steps based on the method of Laplace transform and leads the reader from stochastic process to closed form solution. However, many of the algebraic steps are missing. Here we present the complete solution filling in all the necessary algebraic details.

We believe that this detailed solution will be a valuable resource to the students of option pricing. Students in introductory courses on derivatives may find it helpful in completing their understanding of the problem. It will help the more advanced students in developing solutions to other derivative pricing problems. For those students wishing to pursue careers in financial engineering or financial modeling a working knowledge of financial mathematics is critical. This exercise, developed around one of the foundations of option pricing theory calls on students to engage in the detailed work necessary to solve a continuous time finance problem. The benefit of working in the context of a familiar model allows students to see the steps needed to derive this type of model.

This work can be used as a supplement to Shimko (1992), or as a stand-alone presentation in classes on derivatives, options and futures, or financial engineering involving a more mathematical treatment of the theory. There are several potential classroom applications for this work:

1. An in class presentation delivered by the instructor,
2. A solution to an out-of-class exercise, or
3. A handout for self-directed study.

2 The Partial Differential Equation

Assume that the stock price ($S$) follows a Geometric Brownian motion with mean $\mu$ and variance $\sigma^2$. $t$ denotes time and $z$ is a standard Weiner process:

$$dS = \mu S dt + \sigma S dz$$  

(1)

Assume that the price of a European style call option ($C$) is a function of $S$ and $t$:

$$C = C(S,t)$$  

(2)

By Ito’s Lemma:

$$dC = C_S dS + C_t dt + \frac{1}{2} C_{SS} S^2 dt$$  

(3)

where $C_S$ and $C_t$ are partial derivatives of $C$ with respect to $S$ and $t$, respectively and $C_{SS}$ is the second partial derivative of $C$ with respect to $S$.

Since,

$$dS^2 = \sigma^2 S^2 dz^2 = \sigma^2 S^2 dt$$  

(4)

we can write (3) as:

$$dC = C_S dS + C_t dt + \frac{1}{2} C_{SS} \sigma^2 S^2 dt$$  

(5)

We can form a hedge portfolio by combining $N_S$ shares of stock and $N_C$ call options. The value of the hedge portfolio ($V_H$) is given by:

$$V_H = N_S S + N_C C$$  

(6)

Differentiating (6), we get:

$$dV_H = N_S dS + N_C dC$$  

(7)

Substitute $dC$ from (5) into (7):

$$dV_H = N_S dS + N_C \left( C_S dS + C_t dt + \frac{1}{2} C_{SS} \sigma^2 S^2 dt \right)$$  

(8)

Rearrange terms in (8) to get:

$$dV_H = (N_S + N_C C_S) dS + \left( N_C C_t + N_C \frac{1}{2} C_{SS} \sigma^2 S^2 \right) dt$$  

(9)

A perfectly hedged portfolio should be risk free and earn a risk free rate. Therefore,

$$dV_H = r V_H dt$$
Substitute this $dV_H$ into (9):

$$rV_H dt = (N_S + N_C C_S) dS + \left( N_C C_t + N_C \frac{1}{2} C_{SS} \sigma^2 S^2 \right) dt \quad (10)$$

So $N_S + N_C C_S = 0$ to make both sides of (10) deterministic. So that,

$$rV_H dt = \left( N_C C_t + N_C \frac{1}{2} C_{SS} \sigma^2 S^2 \right) dt \quad (11)$$

Dividing by $dt$ and substituting (6) for $V_H$

$$rNS + rNC = \left( N_C C_t + N_C \frac{1}{2} C_{SS} \sigma^2 S^2 \right) \quad (12)$$

Rearranging and dividing by $NC$

$$C_t = r \frac{N_S}{N_C} S + rC - \frac{1}{2} C_{SS} \sigma^2 S^2 \quad (13)$$

Now we define $\tau = T - t$ where $t$ is the chronological time, $T$ is the time at which the option matures, and $\tau$ is the time remaining to maturity. Then, $C_t = -C_{\tau}$. Also, let $N_S = 1$, then since $N_S + N_C C_S = 0$, $N_C = \frac{-1}{C_S}$. Substituting for $C_t$, $N_S$, and $N_C$ and rearranging, we get:

$$\frac{1}{2} \sigma^2 S^2 C_{SS} + rSC_S - rC - C_{\tau} = 0 \quad (14)$$

The initial and boundary conditions for (14) are

$$C(S, 0) = \text{Max}(S - E, 0) \quad (14a)$$

$$C(0, \tau) = 0 \quad (14b)$$

Initial condition (14a) specifies the option value upon maturity while boundary condition (14b) states that the option is worthless when the stock price is zero.

3 The Laplace Transform

If we let $g(S) = L_q[C(S, \tau)]$, where $L_q$ is the Laplace operator and $g$ is the Laplace transform with parameter $q$ then $L_q[C_{\tau}(S, \tau)] = qL_q[C(S, \tau)] - C(S, 0) = qL_q[C(S, \tau)] - \text{Max}(S - E, 0)$

Taking Laplace transform of equation (14) and using appropriate substitutions, (14) becomes

$$\frac{1}{2} \sigma^2 S^2 g_{SS} + rgg_S - [tg - \text{Max}(S - E, 0)] = 0 \quad (15)$$

Rearranging, we get

$$\frac{1}{2} \sigma^2 S^2 g_{SS} + rgg_S - (r + q)g = -\text{Max}(S - E, 0) \quad (16)$$
We will have two equations for the Laplace transform $g$ in equation (15). One when $S \geq E$ and the other one when $S \leq E$. The homogenous equation in both cases is:

$$\frac{1}{2} \sigma^2 S^2 g_{SS} + rSg_S - (r + q)g = 0 \quad (17)$$

We will solve the homogenous equation to get the general solution. For $S \geq E$ case, we also have a particular solution because the differential equation is nonhomogeneous.

Assume that the solution to (17) is of the form $g = AS^\gamma$ so $g_S = A\gamma S^\gamma - 1$ and $g_{SS} = A(\gamma^2 - \gamma)S^{\gamma - 2}$. Substituting into (17), we get:

$$A \frac{1}{2} \sigma^2 S^2(\gamma^2 - \gamma)S^{\gamma - 2} + Ar\gamma S^{\gamma - 1} - A(r + q)S^\gamma = 0 \quad (18)$$

This can be factored as:

$$AS^{\gamma} \left[ \frac{1}{2} \sigma^2 \gamma^2 + \left( r - \frac{1}{2} \sigma^2 \right) \gamma - (r + q) \right] = 0 \quad (19)$$

For a non trivial solution, $A$ should be nonzero. Therefore, the terms in $[\cdots]$ must be zero. This leads to a quadratic equation whose roots are:

$$\gamma = \frac{-(r - \frac{1}{2} \sigma^2) \pm \sqrt{(r - \frac{1}{2} \sigma^2)^2 + (2\sigma^2)(r + q)}}{\sigma^2} \quad (20)$$

Since we have two distinct roots, the solution to (17) is:

$$g = A_1 S^{\gamma_1} + A_2 S^{\gamma_2} \quad (21)$$

where,

$$\gamma_1 = \frac{-(r - \frac{1}{2} \sigma^2) + \sqrt{(r - \frac{1}{2} \sigma^2)^2 + 2\sigma^2(r + q)}}{\sigma^2} \quad (21a)$$

$$\gamma_2 = \frac{-(r - \frac{1}{2} \sigma^2) - \sqrt{(r - \frac{1}{2} \sigma^2)^2 + 2\sigma^2(r + q)}}{\sigma^2} \quad (21b)$$

Note that $\gamma_1 \geq 0 \geq \gamma_2$.

It will be useful to note that:

$$\gamma_1 - \gamma_2 = \frac{2\sqrt{(r - \frac{1}{2} \sigma^2)^2 + 2\sigma^2(r + q)}}{\sigma^2} \quad (21c)$$

$$\gamma_1 - 1 = \frac{-(r + \frac{1}{2} \sigma^2) + \sqrt{(r - \frac{1}{2} \sigma^2)^2 + 2\sigma^2(r + q)}}{\sigma^2} \quad (21d)$$

$$\gamma_2 - 1 = \frac{-(r + \frac{1}{2} \sigma^2) - \sqrt{(r - \frac{1}{2} \sigma^2)^2 + 2\sigma^2(r + q)}}{\sigma^2} \quad (21e)$$
For $S \leq E$ case, (21) is the complete solution. For the case $S \geq E$, we must solve for the particular solution as well. The equation for $S \geq E$ case is:

$$A S^\gamma \left[ \frac{1}{2} \sigma^2 \gamma^2 + \left( r - \frac{1}{2} \sigma^2 \right) \gamma - (r + q) \right] = -(S - E)$$

(22)

Assume $g = mS + n$ so $g_S = m$ and $g_{SS} = 0$. Upon substitution, we get

$$rSm - (r + q)(mS + n) = -(S - E)$$

(23)

Rearranging, we get:

$$-qSm - (r + q)n = -(S - E)$$

(24)

or,

$$qSm + (r + q)n = S - E$$

(25)

Matching the coefficients of terms with $S$ and without $S$, we get

$$m = \frac{1}{q} \quad \text{and} \quad n = -\frac{E}{(r + q)}$$

Therefore, for the case $S \geq E$, the solution is:

$$g = A_1 S^{\gamma_1} + A_2 S^{\gamma_2} + \frac{S}{q} \frac{E}{(r + q)}$$

(26)

Recall that $\gamma_1 \geq 0 \geq \gamma_2$. In the case of $S \geq E$, $A_1 = 0$ to ensure the boundedness of the derivative $g_S$. In the case of $S \leq E$, $A_2 = 0$ to ensure that the option’s value approaches zero as the stock price goes to zero. The solutions for the two cases (equations (21) and (26)), then reduce to:

$$g(S \geq E) = A_2 S^{\gamma_2} + \frac{S}{q} \frac{E}{(r + q)}$$

(27a)

$$g(S \leq E) = A_1 S^{\gamma_1}$$

(27b)

We require the option pricing function to be continuous and differentiable at the transition point $S = E$. Therefore, the values of the function and their first derivatives from (27a) and (27b) must equal each other. These conditions can be used to solve for $A_1$ and $A_2$. The function values and derivatives at $S = E$ are:

$$g(S \geq E) \bigg|_E = A_2 E^{\gamma_2} + \frac{E}{q} \frac{E}{(r + q)}$$

(28a)

$$g(S \leq E) \bigg|_E = A_1 E^{\gamma_1}$$

(28b)

$$g'(S \geq E) \bigg|_E = A_2 \gamma_2 E^{\gamma_2 - 1} + \frac{1}{q}$$

(29a)

$$g'(S \leq E) \bigg|_E = A_1 \gamma_1 E^{\gamma_1 - 1}$$

(29b)
where $E$ denotes the value at $S = E$.

By setting (28a)=(28b) and (29a)=(29b), we get two equations for the two unknowns $A_1$ and $A_2$:

\[ A_2 E^{\gamma_2} + \frac{E}{q} \frac{E}{q + r + q} = A_1 E^{\gamma_1} \quad (30a) \]
\[ A_2 \gamma_2 E^{\gamma_2 - 1} + \frac{1}{q} = A_1 \gamma_1 E^{\gamma_1 - 1} \quad (30b) \]

Multiply (30a) by $\frac{\gamma_1}{E}$ and subtract (30b) from it to get:

\[ A_2 (\gamma_1 - \gamma_2) E^{\gamma_2 - 1} + \frac{\gamma_1}{q} - \frac{\gamma_1}{r + q} - \frac{1}{q} = 0 \quad (31) \]

Solving for $A_2$, we get:

\[ A_2 = \left[ \frac{\gamma_1}{r + q} - \frac{(\gamma_1 - 1)}{q} \right] \frac{E^{1 - \gamma_2}}{(\gamma_1 - \gamma_2)} \quad (32) \]

Substituting for $A_2$ in (30b), we get $A_1$:

\[ A_1 = \left[ \frac{\gamma_2}{r + q} - \frac{(\gamma_2 - 1)}{q} \right] \frac{E^{1 - \gamma_1}}{(\gamma_1 - \gamma_2)} \quad (33) \]

Now we have solved for the Laplace transform $g$. The problem, therefore, is reduced to taking the inverse Laplace transform of $g$. There are two $g$’s, one for $S \geq E$ and the other for $S \leq E$. The inverse Laplace transforms of both are the same. We will take the inverse Laplace transform of (27b) which can be written as,

\[ g(S \leq E) = \left[ \frac{\gamma_2}{q + r} - \frac{(\gamma_2 - 1)}{q} \right] \frac{E^{1 - \gamma_1}}{(\gamma_1 - \gamma_2)} S^{\gamma_1} \quad (34) \]

4 The Inverse Laplace Transform

To get the solution to look like the familiar Black-Scholes formula, the following Laplace transform is going to be useful:

\[ L_q \left[ e^{\tau N} \left( \frac{d + b \tau}{c \sqrt{\tau}} \right) \right] = \frac{1}{2} e^{-ak} \frac{e^{-k \sqrt{q-f+a^2}}}{\sqrt{q-f+a^2}} \frac{e^{-k \sqrt{q-f+a^2}}}{\sqrt{q-f+a^2}} \]

where $a = -\frac{b}{c \sqrt{q}}$, and $k = -\frac{a}{c \sqrt{q}}$, and $b, c, d, f$ are arbitrary constants.

Now, our efforts will be concentrated on rearranging (34) so that the useful Laplace transform may be applied. First, we separate it out into two terms as:

\[ \frac{\gamma_2}{q + r} \frac{E^{1 - \gamma_1}}{(\gamma_1 - \gamma_2)} S^{\gamma_1} \quad (34a) \]
\[ -\frac{(\gamma_2 - 1)}{q} \frac{E^{1 - \gamma_1}}{(\gamma_1 - \gamma_2)} S^{\gamma_1} \quad (34b) \]
Performing some algebra on (34a) term, It can be written as:

\[
E \left( \frac{\gamma_2}{q + r} \right) \left( \frac{1}{\gamma_1 - \gamma_2} \right) \left( \frac{S}{E} \right)^{\gamma_1} \tag{35}
\]

Noting that

\[
\left( \frac{S}{E} \right)^{\gamma_1} = e^{\gamma_1 \ln \left( \frac{S}{E} \right)}
\]

and substituting for \( \gamma_2 \) from (21b) and \( \gamma_1 - \gamma_2 \) from (21c), (35) can be written as:

\[
-E \left( \frac{\gamma_2}{q + r} \right) \left( \frac{1}{\gamma_1 - \gamma_2} \right) \left( \frac{1}{\sigma^2} \right) \left( \frac{1}{2\sqrt{(r - \frac{1}{2}\sigma^2)^2 + 2\sigma^2(r + q)}} \right) e^{\gamma_1 \ln \left( \frac{S}{E} \right)} \tag{36}
\]

Multiplying and dividing by \( -(r - \frac{1}{2}\sigma^2)^2 + 2\sigma^2(r + q) \) and using the identity \((x + y)(x - y) \equiv x^2 - y^2\), we get:

\[
-E \left[ \frac{-(r - \frac{1}{2}\sigma^2)^2 + 2\sigma^2(r + q)}{(r + q)\sqrt{(r - \frac{1}{2}\sigma^2)^2 + 2\sigma^2(r + q)}} \right] \frac{1}{2} e^{\gamma_1 \ln \left( \frac{S}{E} \right)} \tag{37}
\]

This can be simplified to:

\[
-E \left[ \frac{1}{\frac{1}{\sigma^2} \sqrt{(r - \frac{1}{2}\sigma^2)^2 + 2\sigma^2(r + q)}} \right] \frac{1}{2} e^{\gamma_1 \ln \left( \frac{S}{E} \right)} \tag{38}
\]

Substituting for \( \gamma_1 \) from (21a) and using the identity \( e^{x+y} \equiv e^x e^y \), we get:

\[
-E \left[ \frac{1}{\frac{1}{\sigma^2} \sqrt{(r - \frac{1}{2}\sigma^2)^2 + 2\sigma^2(r + q)}} \right] \frac{1}{2} e \left( \frac{-(r - \frac{1}{2}\sigma^2)}{\sigma^2 \ln \left( \frac{S}{E} \right)} \right) \left( \frac{\ln \left( \frac{S}{E} \right)}{\ln \left( \frac{S}{E} \right) \gamma_1 \sigma^2} \right) \left( \frac{\sqrt{(r - \frac{1}{2}\sigma^2)^2 + 2\sigma^2(r + q)}}{\sqrt{(r - \frac{1}{2}\sigma^2)^2 + 2\sigma^2(r + q)}} \right) \tag{39}
\]

This can be written as:

\[
-E \left[ \frac{1}{\frac{1}{\sigma^2} \sqrt{(r - \frac{1}{2}\sigma^2)^2 + 2\sigma^2(r + q)}} \right] \frac{1}{2} e \left( \frac{-(r - \frac{1}{2}\sigma^2)}{\sigma^2 \ln \left( \frac{S}{E} \right)} \right) \left( \frac{\ln \left( \frac{S}{E} \right)}{\ln \left( \frac{S}{E} \right) \gamma_1 \sigma^2} \right) \left( \frac{\sqrt{(r - \frac{1}{2}\sigma^2)^2 + 2\sigma^2(r + q)}}{\sqrt{(r - \frac{1}{2}\sigma^2)^2 + 2\sigma^2(r + q)}} \right) \tag{40}
\]
Taking the inverse Laplace transform of (40), we get our first term:

$$-Ee^{-\tau N} \left[ \frac{\ln \left( \frac{S}{T} \right) + (r - \frac{1}{2}\sigma^2)\tau}{\sigma \sqrt{\tau}} \right] \quad (I)$$

Now, we’ll work on (34b) to get the second term. Write (34b) as:

$$-(\gamma_2 - 1) \left( \frac{S}{\gamma_1 - \gamma_2} \right) \left( \frac{S}{E} \right)^{\gamma_1 - 1}$$

Noting that

$$\left( \frac{S}{E} \right)^{\gamma_1} = e^{\gamma_1 \ln \left( \frac{S}{E} \right)}$$

and substituting for $\gamma_1 - \gamma_2$ from (21c), (41) can be written as:

$$-\left( \frac{S}{q} \right) \frac{(\gamma_2 - 1)}{2\sqrt{(r - \frac{1}{2}\sigma^2)^2 + 2\sigma^2(r + q)}} e^{(\gamma_1 - 1)\ln \left( \frac{q}{S} \right)} \quad (42)$$

Substituting for $\gamma_2 - 1$ from (21b), we get:

$$-\left( \frac{S}{q} \right) \frac{-(r + \frac{1}{2}\sigma^2) - \sqrt{(r - \frac{1}{2}\sigma^2)^2 + 2\sigma^2(r + q)}}{2\sqrt{(r - \frac{1}{2}\sigma^2)^2 + 2\sigma^2(r + q)}} e^{(\gamma_1 - 1)\ln \left( \frac{q}{S} \right)} \quad (43)$$

which simplifies to,

$$\left( \frac{S}{q} \right) \left( r + \frac{1}{2}\sigma^2 \right) + \sqrt{(r - \frac{1}{2}\sigma^2)^2 + 2\sigma^2(r + q)} \frac{1}{2} e^{(\gamma_1 - 1)\ln \left( \frac{q}{S} \right)} \quad (44)$$

Multiplying and dividing (44) by $-(r + \frac{1}{2}\sigma^2) + \sqrt{(r - \frac{1}{2}\sigma^2)^2 + 2\sigma^2(r + q)}$ and using the identity $(x + y)(x - y) \equiv x^2 - y^2$ we get:

$$\left( \frac{S}{q} \right) \frac{-(r + \frac{1}{2}\sigma^2)^2 + (r - \frac{1}{2}\sigma^2)^2 + 2\sigma^2(r + q)}{\sqrt{(r - \frac{1}{2}\sigma^2)^2 + 2\sigma^2(r + q)}} \left[ -(r + \frac{1}{2}\sigma^2) + \sqrt{(r - \frac{1}{2}\sigma^2)^2 + 2\sigma^2(r + q)} \right] \frac{1}{2} e^{(\gamma_1 - 1)\ln \left( \frac{q}{S} \right)} \quad (45)$$

Expanding the numerator, we get $-(r^2 + \sigma^2 r + \frac{1}{4}\sigma^4) + (r^2 - \sigma^2 r + \frac{1}{4}\sigma^4) + 2\sigma^2 r + 2\sigma^2 q = 2\sigma^2 q$. So, we can write (45) as:

$$\left( \frac{S}{q} \right) \frac{2\sigma^2 q}{\sqrt{(r - \frac{1}{2}\sigma^2)^2 + 2\sigma^2(r + q)}} \left[ -(r + \frac{1}{2}\sigma^2) + \sqrt{(r - \frac{1}{2}\sigma^2)^2 + 2\sigma^2(r + q)} \right] \frac{1}{2} e^{(\gamma_1 - 1)\ln \left( \frac{q}{S} \right)} \quad (46)$$
Taking the inverse Laplace transform of (50), we get our second term:

\[
S \frac{1}{2\sigma} \sqrt{(r - \frac{1}{2}\sigma^2)^2 + 2\sigma^2(r + q)} \cdot \frac{1}{\sigma^2} \left[ -r + \frac{1}{2}\sigma^2 + \sqrt{(r - \frac{1}{2}\sigma^2)^2 + 2\sigma^2(r + q)} \right] e^{\frac{1}{2}((\gamma_1 - 1)\ln \left( \frac{S}{\tau} \right))}
\]

Substituting for \(\gamma_1\) from (21d), doing some algebra, and using the identity \(e^{x+y} = e^x e^y\), we get:

\[
S \frac{1}{e} \left( \frac{-(r + \frac{1}{2}\sigma^2)\ln \left( \frac{S}{\tau} \right)}{\sigma^2} \right) e \left( \frac{\sqrt{(r - \frac{1}{2}\sigma^2)^2 + 2\sigma^2(r + q)}}{2\sigma^2} \ln \left( \frac{S}{\tau} \right) \right)
\]

Doing some rearranging,

\[
S \frac{1}{2e} \left( \frac{-(r + \frac{1}{2}\sigma^2)\ln \left( \frac{S}{\tau} \right)}{\sigma^2} \right) e \left( \frac{\sqrt{(r - \frac{1}{2}\sigma^2)^2 + 2\sigma^2(r + q)}}{2\sigma^2} \ln \left( \frac{S}{\tau} \right) \right)
\]

Expanding the terms under the radical, we get:

\[
\frac{r^2 - \sigma^2 r + \frac{1}{4}\sigma^4}{2\sigma^2} + r + q
\]

\[
\Rightarrow \frac{r^2 + \sigma^2 r + \frac{1}{4}\sigma^4}{2\sigma^2} + q
\]

\[
\Rightarrow \frac{(r + \frac{1}{2}\sigma^2)^2}{2\sigma^2} + q
\]

Making this substitution, and doing some rearranging, we get:

\[
S \frac{1}{2e} \left( \frac{-(r + \frac{1}{2}\sigma^2)\ln \left( \frac{S}{\tau} \right)}{\sigma^2} \right) e \left( \frac{\ln \left( \frac{S}{\tau} \right)\sigma^2}{\sqrt{2}} + r + q \right)
\]

Again, this matches our useful Laplace form with \(b = (r + \frac{1}{2}\sigma^2), c = \sigma, d = \ln \left( \frac{S}{\tau} \right)\), and \(f = 0\), so that \(a = -\frac{b}{c\sqrt{2}} = -\frac{(r + \frac{1}{2}\sigma^2)}{\sigma\sqrt{2}}\) and \(k = -\frac{d\sqrt{2}}{c} = -\frac{\ln \left( \frac{S}{\tau} \right)\sqrt{2}}{\sigma}\). Taking the inverse Laplace transform of (50), we get our second term:

\[
SN \left[ \frac{\ln \left( \frac{S}{\tau} \right) + (r + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{2}} \right]
\]
Combining the two terms \([I]\) and \([II]\), we get our solution:

\[
C = SN(d_1) - e^{-r\tau}N(d_2)
\]

where

\[
d_1 = \frac{\ln \left( \frac{S}{F} \right) + \left( r + \frac{1}{2} \sigma^2 \right) \tau}{\sigma \sqrt{\tau}}
\]

\[
d_2 = \frac{\ln \left( \frac{S}{F} \right) + \left( r - \frac{1}{2} \sigma^2 \right) \tau}{\sigma \sqrt{\tau}} = d_1 - \sigma \sqrt{\tau}
\]

5 Conclusion

In this work we present a complete solution to the Black-Scholes equation. This work is appropriate for students in more advance classes in derivatives, options and futures, or financial engineering in which there is a heavy reliance on continuous time financial mathematics.

References


